Design Of Active Butterworth Filters

Part 1 of an article by Dr G Brown, B.Sc, Ph.D, C.Eng, FIEE, G1VCY

BUTTERWORTH FILTERS have a long and distinguished history in electronic engineering. Until recently, they were composed of passive devices - inductors, capacitors and resistors. However, the increasing need to implement filters with easily-controlled characteristics at lower frequencies has caused the passive realisation to give way to the active version, using operational amplifiers.

Although the active filter does give easily-controlled characteristics, it is perhaps more important to the designer that the use of inductors is no longer necessary. This is doubly attractive at lower frequencies, where the size of the inductor (both in value and in physical dimensions) becomes problematic. This article provides a simple guide to the design of active Butterworth filters up to 8th order, emphasising the building-block approach by cascading stages of first and second order, due consideration being given to the calculation of stage gain to achieve the required characteristics.

Sufficient detail is given in the text upon which to base a simple computer programme. Calculations can be performed easily on a hand calculator, but repetitive use of the same functions cries out for a programme to be written. I can vouch for the fact that the time spent in programme-writing is well worthwhile, particularly for 'what if' investigations, provided that they are not used as replacements for the thought process!

FILTER CHARACTERISTICS

A SIMPLE FILTER HAS three descriptive parameters. A pass-band is the region where input frequencies pass through unaltered to the output. A stop-band describes the region where input frequencies are prevented from reaching the output. A cut-off frequency represents the boundary between the two bands. An ideal low-pass filter response is shown in **Fig 1**.

The pass-band is on the low-frequency side of the cut-off, the stop-band being above it. In this case, the response is rectangular and the identification of the cut-off frequency is trivial. The response of a practical filter is also shown on the same axes. Although exhibiting the same general features as the ideal filter, it does differ in the following ways.

- The pass-band is not flat-topped.
- The stop-band is not flat-bottomed.
- The cut-off line is not vertical, but merges into both the stop-band and pass-band.
- The stop-band is not infinitely attenuating, but has finite attenuation, and is asymptotic [see Glossary of Terms] to the zero response axis, ie the output is zero only at infinite frequency.

This discrepancy between theory and practice does not surprise most people, even if the reasons are somewhat deep-rooted in mathematics. What concerns us here is that it is the real filter that we have to live with, and thus we must know how to work with and understand the imperfect shape. To this end, it may seem surprising that, in the design of Butterworth filters for general-purpose use, the only parameters of basic interest are the cut-off frequency and the steepness of the cut-off. This argument applies equally to low-pass and high-pass filters.

AREAS OF APPLICATION

CONSIDER NOW SOME AREAS of interest within amateur radio where the use of low-frequency active filters is invaluable. Everyone will doubtless have his own pet area in which filters could be useful or even vital, but the areas chosen for illustration are Radio Teletype (RTTY), Slow-Scan Television (SSTV) and Morse (CW).

RTTY

For amateur use, the recommended tones are 1275Hz and 1445Hz, for a shift frequency of 170Hz. The pass-band of the average amateur receiver extends from 300Hz to 2700Hz, so at least 1255Hz of receiver bandwidth above the upper tone and 975Hz below the lower tone are responsible for passing unwanted signals and noise to the terminal unit. Put another way, the receiver is producing and receiving noise over a bandwidth of 2400Hz, only 170Hz of which contains wanted signal components, therefore, the amount of noise being passed to the terminal unit is 2400/170 = 14 times larger than it need be! These figures are very approximate, because they ignore the information bandwidth of the RTTY signal, but serve to indicate the severity of the problem and the need for a good filter.

As was mentioned earlier, an ideal rectangular filter function is impossible to achieve, so the choice of steepness of cut-off becomes a matter of knowing what your RTTY terminal unit can tolerate in terms of out-of-band interference. The equations presented here will make the design decisions fairly straight forward once the magnitude of a particular application problem has been assessed. The preceding discussion assumed that a band-pass filter was to be used, one where the pass-band is sandwiched between two stop-bands. Such a filter is easily implemented by cascading (ie connecting one after the other) low-pass and high-pass sections, as illustrated in **Fig 2**. Note, however, that the pass-band is defined by two frequencies, f1 and f2, with f2>f1; the cut-off of the high-pass section must occur at the lower frequency, and that of the low-pass section must occur at the higher frequency. If this basic design criterion is forgotten, the resulting filter will be of the all-stop variety!

SSTV

Similar considerations apply here, but whereas a RTTY signal consists of two very narrow-band signals, the SSTV signal exists over an 1100Hz band between sync bottoms at 1200Hz and peak white at 2300Hz. Because this signal requires a bandwidth which is narrower that the 2400Hz receiver band-

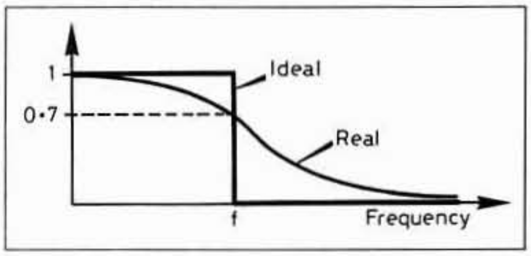

Fig 1: Practical filters will differ from the ideal case.

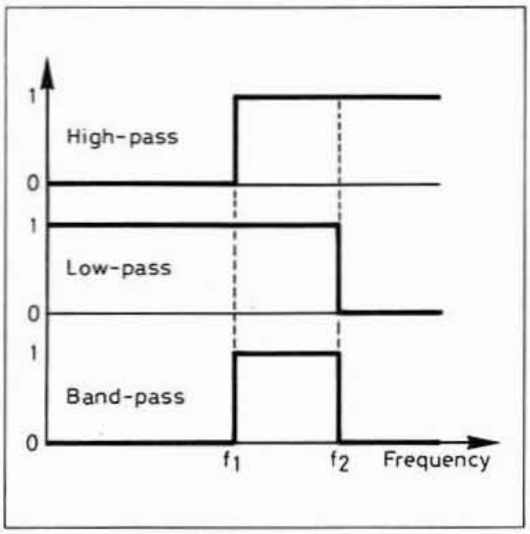

Fig 2: Band-pass filter made from low and high-pass filters.

width, much of the available bandwidth is therefore contributing noise and unwanted signals. (This argument applies to the transmission and reception of SSTV by single sideband, as used on the HF bands, but not to the same extent when frequency modulation is used, as on VHF and UHF.) A band-pass filter is again needed to prevent these extraneous signals reaching the SSTV decoder.

There are other areas within SSTV where filtering is advisable, but often overlooked. An example of this is the provision of an anti-aliasing filter. Aliasing is the name given to the production of jagged edges in a picture which consists of discrete picture elements, or pixels. A typical SSTV picture consists of 128 x 128 pixels, and any line which is not exactly vertical or horizontal may be displayed with a jagged edge. An anti-aliasing filter is a low-pass filter designed to reduce this effect, and is placed in the transmitting equipment before the analogue-to-digital converter (ADC). The explanation of this is outside the scope of this article, but the golden rule, whenever an ADC is used, is never to pass to it any signal frequencies higher than half the sampling (clock) frequency. Whether or not a particular system will benefit markedly from an anti-aliasing filter can be seen by defocusing the camera slightly. Most pictures will show an improvement, and some will be enhanced enormously.

In the receiving system, a low-pass filter should be used after the digital-to analogue converter (DAC). Most simple SSTV designs use a resistive DAC, and produce a fast-scan output resembling a chequer board. This is because the fast transitions between individual pixels are being faithfully passed by the slow- to- fast scan converter. A low-pass filter designed with a cut-off at around half the clock frequency will blur these transitions without noticeably degrading the picture.

Low-pass filters having the correct characteristics can easily be incorporated into existing designs, and the experimenter will find little difficulty in evaluating the cut-off frequencies required.

CW

Having got to this stage, there is little to say in terms of the application of a good band-pass filter to the decoding of CW transmissions, either aurally or automatically. The pass-band for CW depends to a great extent on the performance of the automatic decoder or the preference of the human operator. Typically, frequencies over a range of 250 to 500 Hz cover most requirements. The steepness of the two cut-off frequencies is very important here, and cascading two high-order Butterworth filters may not be the best solution. The reader is referred to an excellent article by G0CKZ in *Radcom* [1].

In concluding this section on typical applications of active filters, it must be stated that the best filtering (in most cases) should take place in the receiver IF stages, and active filters of the type under discussion are intended for those who find their receiver selectivity performance less than adequate, or who do not have any form of IF shift and width controls.

GLOSSARY OF TERMS

Asymptotic

Imagine the curve $y = 1/x$. As x increases, y decreases. As x approaches infinity, y approaches, but never reaches zero (the x axis). In such a case we say that y is asymptotic to the x axis.

Coefficient

In an expression such as;

$$Ax^3 + Bx^2 + Cx + D = 0$$

A, B and C are coefficients, ie constants which multiply the variable.

In-Band

This is a term used to describe a frequency (or frequencies) lying within the pass-band of any filter. For a low-pass filter, in-band frequencies lie below the cut-off, and for a high-pass filter they lie above the cut-off.

Polynomial

A polynomial is an algebraic expression containing more than one term. For example;

$x^2 + 4x + 2$ is a polynomial of order 2 (the highest power of x).

Quadratic

(a) Equation. A quadratic equation is a polynomial of order 2, eg;

$$x^2 + 5x + 6 = 0$$

(b) Factor .The above equation can be split into factors which, when multiplied together, are equal to the original expression.

For example $(x + 3)$ and $(x + 2)$ are the factors of the previous equation, because

$$(x + 3)(x + 2) = x^2 + 5x + 6$$

REFERENCE

[1] Active eliptic audio filter design using op-amps, D H G Fritsch, G0CKZ, *RadCom*, Feb/Mar 1986

. . . to be continued

RADCOM TECHNICAL FEATURE

Design Of Active Butterworth Filters

Part 2 of an article by Dr G Brown, B.Sc, PhD, C.Eng, FIEE, G1VCY

THE APPENDIX PROVIDES sufficient of the theory to permit an understanding of the derivation of component values. This section quotes results from the Appendix, where relevant, to illustrate the factors governing component choice.

Before plunging into the filter design, it may be useful to address the obvious question "Why a Butterworth filter, and what does its 'order' signify?" There are two well-known filter types, Butterworth and Chebyshev. Both are based upon different mathematical polynomials (see Appendix), the behaviour of which can be produced by a handful of components and an operational amplifier. In a nutshell, a Butterworth filter is better when the application demands a flat pass-band, good impedance matching, and a well-behaved phase-shift characteristic. A Chebyshev design is capable of a sharper cut-off, but exhibits a well-known ripple in the passband, together with some impedance mismatch. There are also other differences which need not be considered at this stage.

The 'order' of a filter is defined as the value of the exponent n in the Butterworth Polynomial (see Appendix). In practical terms, it governs the steepness of the cut-off between the pass and stop regions of the characteristic. Thus, a second-order filter has a steeper cut-off than a first-order, and so on.

Fig 3 shows the basic circuit of a first-order active Butterworth filter. As can be seen, this is simply an operational amplifier in a non-inverting, high input impedance configuration, with a simple resistor/capacitor (RC) low-pass filter at its input. The second-order stage is shown in **Fig 4**, and may be recognised as a voltage-controlled voltage source (VCVS) circuit. Again, the operational amplifier is configured as a high input impedance device, but the initial RC filter is modified and now includes frequency-dependent feedback from the output.

Two points of commonality are worth noting. Firstly, the gain is set by the resistors R_a and R_b in Fig 3 and by R_a and R_i in Fig 4. For simplicity, R_a can have the same value in both circuits (although this is not obligatory), the gain then being set by the choice of R_b and R_i, as will be shown. Secondly, the cut-off frequency is determined by R_t and C, and is the same for all stages of the filter, irrespective of its order. Again, there is no need for these to be the same in the two circuits, or even for the two resistor-capacitor pairs to be the same in Fig 4 but the calculations are much simpler when this is so, and no loss of performance is incurred.

Although the gain and the cut-off frequency are independently controllable, the gain is not the same in each stage of a cascaded design. In contrast, the gain of a first-order stage is arbitrary (within certain limits), and can be chosen to suit requirements.

In both Fig 3 and Fig 4, the feedback network defining the gain is common, and the value of Ra has also been made common to simplify calculations. In Fig 3, the gain G, at zero frequency, is given by

$$\text{Gain} = (R_a + R_b)/R_a$$

and in Fig 4 by

$$\text{Gain} = (R_a + R_i)/R_a$$

For the first-order stage, R_b (and hence G) can be any reasonable value. R_a is chosen to be 10kΩ for the purposes of these illustrations, and thus a value of 10kΩ for R_b would give a stage gain of two (6 dB).

For the second-order stage, the gain must be set to give the correct steepness of cut-off for the order specified. The Appendix shows what the gains of second-order stages must be in a single or cascaded filter. These stage gains, A(i), are given by:

$$A(i) = 3 - K(i)$$

where i is the number of the second-order stage, and K(i) is a coefficient [see Glossary of Terms, in part one] of s as given in **Table 1**.

For example:

A sixth-order filter requires three cascaded second-order stages:

For the first stage,
i = 1 and K(i) = 0.518

For the second stage,
i = 2 and K(i) = 1.414

For the final stage,
i = 3 and K(i) = 1.932

The overall gain is thus:

$$\text{Gain} = (3 - 0.518) * (3 - 1.414) * (3 - 1.932) = 4.2, \text{ or } 12.5 \text{ dB.}$$

Incorporating this into the equation for the second-order gain given above, the value of R_i (i.e. the feedback resistor) in stage (1) becomes:

$$R_i = R_a(2 - K(i))$$

Thus, for the first stage of the above filter:

K(i) = 0.518, giving
R_i = 15kΩ where R_a = 10kΩ.

This is done for each stage of the proposed filter to evaluate the resistances and to determine the stage gains. All that then remains to be done is to derive values for R_t and C to determine the cut-off frequency. Since these values are common to all stages, whether first or second-order, it is performed only once. The equation governing R_t, C and the cut-off frequency f, is:

$$f = 1/(2.\pi.R_t.C)$$

or, expressed in a directly usable form:

$$R_t = 1/(2.\pi.f.C)$$

Notice that C is used as the independent variable here. The reason for this is quite simple. It is much easier to choose a value of C from available components and then calculate the nearest preferred-value resistor to R_t, than vice versa. Even making up odd values of resistance from preferred values is better than doing the same with capacitors.

That is all there is to it, but what has not been fully explained so far, is the high-pass version of the Butterworth filter. Knowing that the only frequency-dependent components are R_t and C gives a clue to the difference between high-pass and low-pass. In a simple RC low-pass filter, the resistor and capacitor are in series, with the output signal being developed across the capacitor, as in Fig 3 and Fig 4. In a high-pass RC filter, the resistor and the capacitor are still in series, but their positions are reversed, the output signal now being developed across the resistor. Thus the only change required, both in Fig 3 and Fig 4, is simply to interchange R_t and C. The calculation of the values is unchanged, as in the example below:

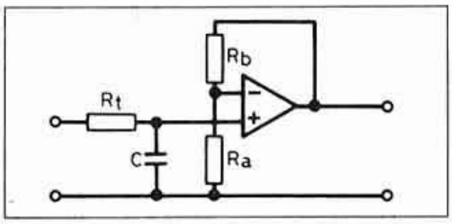

Fig 3: A first order active low-pass filter.

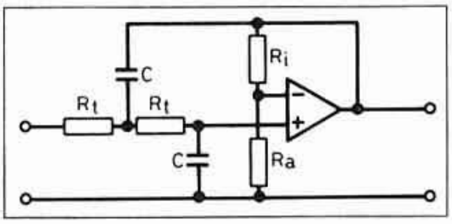

Fig 4: The second order low-pass stage.

Stage 1 of 3

Gain = 3 – 0.518 = 2.842

R_i = (2 – 0.518) * 10000
= 14,800Ω, for R_a = 10kΩ.

Choose C = 10nF.

R_t = 1/(2.π.f.C)
= 13,262Ω, for a cut-off f = 1200Hz.

Stage 2 of 3

Gain = 3 – 1.414 = 1.586

R_i = (2 – 1.414) * 10000
= 5,680Ω, for R_a = 10kΩ.

R_t and C as in stage 1.

Stage 3 of 3

Gain = 3 – 1.932 = 1.068

R_i = (2 – 1.932) * 10000
= 680Ω, for R_a = 10kΩ.

R_t and C as in stages 1 and 2.

Overall Filter Characteristics

Gain = 2.482 * 1.586 * 1.068
= 4.204, or 12.5dB.

Cut-off frequency = 1200Hz.

SUMMARY

THIS ARTICLE HAS GIVEN the design data for Butterworth filters up to eighth order. Some of the background information is presented, so that the design principles can be understood, and three application areas for such filters are described. A worked example has been provided for guidance. The Appendix contains the more indigestible parts of the design process, which should enable those suitably inclined to dig a little deeper into the fascinating world of filters.

APPENDIX

AN APPROXIMATION OF an ideal low-pass filter characteristic is given by:

$$V = 1/P(s)$$

where P(s) is an algebraic expression containing s. Active filters use operational amplifiers or discrete transistors as the active elements, with resistors and capacitors as the only passive elements.

A low-pass filter becomes a Butterworth type when the above equation is approximated by the use of Butterworth Polynomials, Bn(s), where n is the order of the polynomial [see Glossary of Terms in part one];

$$V_{out}(s) = V_{in}(s)/Bn(s)$$

Putting s = j.2.π.f gives;

$$|V_{out}(s)| = |V_{in}(s)|\,|V_{in}(-s)| \text{ where } j=\sqrt{-1}$$

$$= \frac{V_{in}^2}{1+\left(\frac{f}{f_0}\right)^{2n}}$$

From the last two equations, the magnitude of Bn as a function of f can be written as:

$$Bn(f) = \sqrt{1+\left(\frac{f}{f_0}\right)^{2n}}$$

The best way of appreciating the Butterworth response is to normalise the response to filter cut-off frequency, as shown in **Fig 5**. In this way, the response for any cut-off frequency can be mentally scaled from the diagrams shown. For example, a third-order filter with 1200 Hz cut-off, will have a response 18 dB down at 2400 Hz (ie at 2f).

Fig 5 illustrates very clearly that filters of all orders are 3 dB down at the cut-off frequency, and not only does a higher order filter produce a steeper slope, but it allows the passband to be flatter over a marginally wider range.

At this stage, there is a great temptation to work out component values for a first and a second-order stage, and think that cascading the correct number of these will produce a Butterworth filter of the required order. Unfortunately things are not that simple, but don't despair, it is not too difficult. Because it is a polynomial, the eighth-order form is not equivalent to the sum or the product of four second-order forms – it is a separate entity. This means that the polynomials must be known up to the order desired, and the components evaluated accordingly. All that is needed is a table of the polynomials in factorised form, and the required data may be picked out visually. **Table 1** gives the polynomials up to eighth order.

Order n	Polynomial Bn(s)
1	(s+1)
2	(s+1.414s+1)
3	(s+s+1)(s+1)
4	(s+0.765s+1)(s+1.848s+1)
5	(s+0.618s+1)(s+1.618s+1)(s+1)
6	(s+0.518s+1)(s+1.414s+1)(s+1.932s+1)
7	(s+0.445s+1)(s+1.247s+1)(s+1.802s+1)(s+1)
8	(s+0.390s+1)(s+1.111s+1)(s+1.663s+1)(s+1.962s+1)

Table 1. Butterworth Polynomials up to eighth order [1]. See Fig 5.

In the above table, the gain of each second-order stage is related to the coefficients of s in each of the quadratic [see Glossary of Terms in part one] factors (ie in each bracket). The reason for this is irrelevant for the present purpose, as all that is needed is the simple relationship for the in-band [see Glossary of Terms] gain, A, of each second-order stage given by;

$$A = 3 - K$$

where K is the coefficient of s in each quadratic factor (bracket). For example, a second-order filter alone would have a gain of 3 – 1.414 = 1.586, where the figure of 1.414 is the coefficient of s in the bracket on the line of order 2 in the table. Similarly, a fourth-order filter would have two second-order stages cascaded, the first with a gain of 3 – 0.765 = 2.235 and the second with a gain of 3 – 1.848 = 1.152. (From Table 1, order 4 gives 0.765 as the coefficient of s in the first bracket, and 1.848 as the coefficient of s in the second bracket). The gain of any first-order stage is arbitrary, and may be chosen to suit individual requirements.

Fig 5 shows the normalized frequency response for first to eighth order Butterworth filters.

This is all that is needed to specify the gain of separate stages in a cascaded filter. The remaining parameter is the cut-off frequency. This is quite independent of the gain parameters and is given simply by the effective response of a single resistor/capacitor filter as will be seen by referring to the main text.

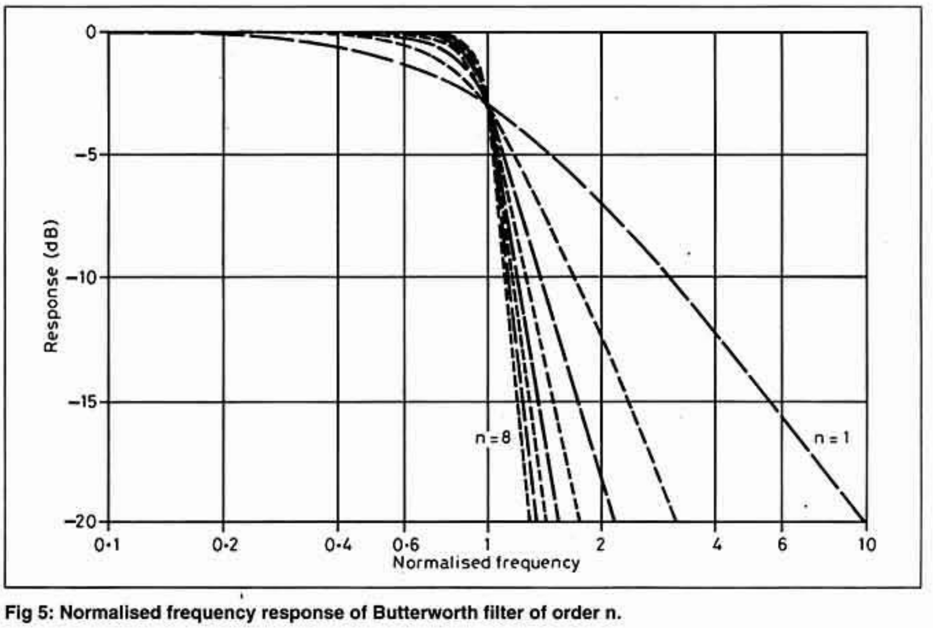

Fig 5: Normalised frequency response of Butterworth filter of order n.

FURTHER READING

[1] *Integrated Electronics* by Millman & Halkias, McGraw-Hill, 1972, pp 548-57.

[2] *Reference Data for Radio Engineers*, Howard W Sams & Co. Inc, 1977, Chapter 10.